On solutions of the reduced model for the dynamical evolution of contact lines

Dmitry Pelinovsky
Department of Mathematics, McMaster University, Hamilton, Ontario, Canada, L8S 4K1

February 7, 2013

Abstract

We solve the linear advection–diffusion equation with a variable speed on a semi-infinite line. The variable speed is determined by an additional condition at the boundary, which models the dynamics of a contact line of a hydrodynamic flow at a 180° contact angle. We use Laplace transform in spatial coordinate and Green's function for the fourth-order diffusion equation to show local existence of solutions of the initial-value problem associated with the set of over-determining boundary conditions. We also analyze the explicit solution in the case of a constant speed (dropping the additional boundary condition).

1 Introduction

Contact lines are defined by the intersection of the rigid and free boundaries of the flow. Flows with the contact line at a 180° contact angle were discussed in [2, 4], where corresponding solutions of the Navier–Stokes equations were shown to have no physical meanings. Recently, a different approach based on the lubrication approximation and thin film equations was developed by Benilov & Vynnycky [1].

As a particularly simple model for the flow shown on Figure 1, the authors of [1] derived the linear advection–diffusion equation for the free boundary $h(x,t)$ of the flow:

$$\frac{\partial h}{\partial t} + \frac{\partial^4 h}{\partial x^4} = V(t)\frac{\partial h}{\partial x}, \quad x > 0, \quad t > 0. \tag{1.1}$$

The contact line is fixed at $x = 0$ in the reference frame moving with the velocity $-V(t)$ and is defined by the boundary conditions $h|_{x=0} = 1$ and $h_x|_{x=0} = 0$. The flux conservation condition is expressed by the boundary condition $h_{xxx}|_{x=0} = -\frac{1}{2}$ (take $\alpha^3 = 3$ in equations (5.12)–(5.13) in [1]).

We assume that $h, h_x, h_{xx} \to 0$ as $x \to \infty$: in fact, any constant value of h at infinity is allowed thanks to the invariance of the linear advection–diffusion equation (1.1) with respect to the shift and scaling transformations. With three boundary conditions at $x = 0$ and the decay conditions as $x \to \infty$, the initial-value problem for equation (1.1) is over-determined and the third (over-determining) boundary condition at $x = 0$ is used to find the dependence of V on t.

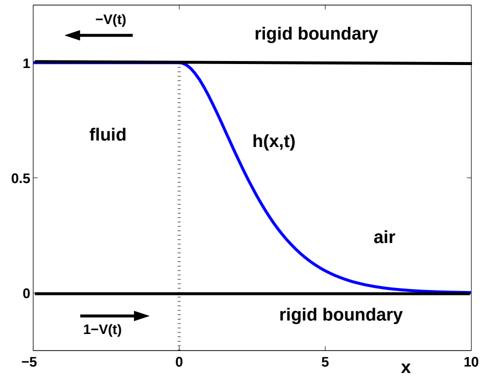

Figure 1: Schematic picture of the flow between rigid boundaries.

We shall consider the initial-value problem with the initial data $h|_{t=0} = h_0(x)$ for a suitable function h_0. In particular, we assume that the profile $h_0(x)$ decays monotonically to zero as $x \to \infty$ and that 0 is a non-degenerate maximum of h_0 such that $h_0(0) = 1$, $h_0'(0) = 0$, and $h_0''(0) < 0$. If the solution $h(x,t)$ losses monotonicity in x during the dynamical evolution, for instance, due to the value of $h_{xx}(0,t)$ crossing 0 from the negative side, then we say that the flow becomes non-physical for further times and the model breaks. Simultaneously, this may mean that the velocity $V(t)$ blows up, as it is defined for sufficiently strong solutions of the advection–diffusion equation (1.1) by the contact equation:

$$h_{xxxxx}(0,t) = V(t) h_{xx}(0,t), \tag{1.2}$$

which follows by differentiation of (1.1) in x and setting $x \to 0$.

The main claim of [1] based on numerical computations of the reduced equation (1.1) as well as more complicated thin-film equations is that for any suitable h_0, there is a finite positive time t_0 such that $V(t) \to -\infty$ and $h_{xx}(0,t) \to -0$ as $t \uparrow t_0$. Moreover, it is claimed that $V(t)$ behaves near the blowup time as the logarithmic function of t, e.g.

$$V(t) \sim C_1 \log(t_0 - t) + C_2, \quad \text{as} \quad t \uparrow t_0, \tag{1.3}$$

where C_1, C_2 are positive constants.

This paper is devoted to analytical studies of solutions of the advection–diffusion equation (1.1) and the effects coming from the inhomogeneous boundary condition $h_{xxx}|_{x=0} = -\frac{1}{2}$ associated with the flux conservation. In particular, we rewrite the evolution equation for the variable $u = h_x$ in the form

$$u_t + u_{xxxx} = V(t) u_x, \quad x > 0, \ \ t > 0, \tag{1.4}$$

subject to the boundary conditions at the contact line

$$u|_{x=0} = 0, \quad u_{xx}|_{x=0} = -\frac{1}{2}, \quad u_{xxx}|_{x=0} = 0, \quad t \geq 0, \tag{1.5}$$

where the boundary conditions $u_{xxx}|_{x=0} = h_{xxxx}|_{x=0} = 0$ follows from the boundary conditions $h|_{x=0} = 1$ and $h_x|_{x=0} = 0$ as well as the original evolution system (1.1) as $x \to 0$.

To simplify the problem, we shall also consider the model for given constant $V(t) = V_0$ and drop the third over-determining boundary conditions at the contact line:

$$\left\{ \begin{array}{ll} u_t + u_{xxxx} = V_0 u_x, & x > 0, \ \ t > 0, \\ u|_{x=0} = 0, & t \geq 0, \\ u_{xx}|_{x=0} = -\frac{1}{2}, & t \geq 0. \end{array} \right. \tag{1.6}$$

Both problems (1.4)–(1.5) and (1.6) are considered under the initial condition $u|_{t=0} = u_0(x)$ with $u_0(0) = 0$, $u_0'(0) < 0$, and $u_0''(0) = -\frac{1}{2}$, as well as the decay condition $u, u_x, u_{xx} \to 0$ as $x \to \infty$.

Using Laplace transform in spatial coordinate and Green's function for the fourth-order diffusion equation, we derive an explicit solution of the boundary-value problem (1.6). In the case $V_0 = 0$, we show that the inhomogeneous boundary condition $h_{xxx}|_{x=0} = u_{xx}|_{x=0} = -\frac{1}{2}$ leads to the secular growth of the boundary value $h_{xx}|_{x=0} = u_x|_{x=0}$ to positive infinity as $t \to \infty$. As a result, even if $h_{xx}|_{x=0} < 0$ initially, the convexity of the solution $h(x,t)$ at the boundary $x = 0$ is lost in the finite time. In the case $V_0 < 0$, we show that no secular growth is observed but the convexity of the solution at the boundary is still lost in the finite time. Applying the same method, we prove local existence of solutions of the original boundary-value problem (1.4)–(1.5). This prepares us to tackle the original conjecture on the finite-time blow-up in the dynamical behavior of the model, which is still left opened for forthcoming studies.

The remainder of this paper is organized as follows. Section 2 reports explicit solutions of the boundary–value problem (1.6) for $V_0 = 0$ and $V_0 \neq 0$. Section 3 gives the local existence result for the original problem (1.4)–(1.5). Appendix A lists properties of Green's function for the fourth-order diffusion equation.

2 Solution for $V(t) = V_0$

Because $V(t)$ is nonconstant for the original problem (1.1), the Laplace transform in time t is not a useful method for this problem. On the other hand, since the boundary-value problem is formulated for half-line, we can use Laplace transform in space x:

$$U(p,t) = \int_0^\infty e^{-px} u(x,t) dx, \quad p > 0. \tag{2.1}$$

We shall develop this method to solve the boundary–value problem (1.6). The explicit solution of this problem will help us to analyze the consequences of inhomogeneous boundary condition $u_{xx}|_{x=0} = -\frac{1}{2}$ and the constant advection term $V(t) = V_0$ on the temporal dynamics of the advection–diffusion equation with the fourth-order diffusion.

Let us denote the boundary values:

$$\beta(t) = u_x|_{x=0}, \quad \gamma(t) = u_{xxx}|_{x=0}. \tag{2.2}$$

Using Laplace transform (2.1), we rewrite an evolution problem associated with the advection–diffusion equation (1.6):

$$\left\{ \begin{array}{l} U_t + p^4 U - V_0 p U = \gamma(t) - \frac{1}{2}p + p^2 \beta(t), \quad t > 0, \\ U|_{t=0} = U_0(p), \end{array} \right. \tag{2.3}$$

where U_0 is the Laplace transform of $u_0 = u|_{t=0}$. By using the variation of parameters, we obtain

$$\begin{aligned} U(p,t) &= U_0(p) e^{-tp^4 + tV_0 p} \\ &\quad + \int_0^t e^{-(t-s)p^4 + (t-s)V_0 p} \left(\gamma(s) - \frac{1}{2}p + p^2 \beta(s) \right) ds. \end{aligned} \tag{2.4}$$

Using the inverse Laplace transform in x, we write this solution in the form:

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px - tp^4 + tV_0 p} \left(\int_0^\infty e^{-py} u_0(y) dy \right) dp \\ &\quad + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} \left(\int_0^t e^{-(t-s)p^4 + (t-s)V_0 p} \left(\gamma(s) - \frac{1}{2}p + p^2 \beta(s) \right) ds \right) dp, \end{aligned} \tag{2.5}$$

where $\mathrm{Re}(c) > 0$ so that the singularities of the integrand in the complex p-plane remain to the left of the contour of integration.

If $t > 0$ is finite, $u_0 \in L^1(\mathbb{R}_+)$, and $\beta, \gamma \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$, Fubini's Theorem implies that the integration in p and in y, s can be interchanged. Let us introduce Green's function $G_t(x)$ for the fourth-order diffusion equation (see Appendix A):

$$G_t(x) = \frac{1}{2\pi i} \int_{c+i\infty}^{c+i\infty} e^{px - tp^4} dp = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-tk^4 + ikx} dk = \frac{1}{\pi} \int_0^\infty e^{-tk^4} \cos(kx) dk.$$

Using Green's function, we can rewrite the solution (2.5) in the implicit form:

$$\begin{aligned} u(x,t) &= \int_0^\infty G_t(x + V_0 t - y) u_0(y) dy - \frac{1}{2} \int_0^t G'_{t-s}(x + V_0(t-s)) ds \\ &\quad + \int_0^t \left[G_{t-s}(x + V_0(t-s)) \gamma(s) + G''_{t-s}(x + V_0(t-s)) \beta(s) \right] ds. \end{aligned} \tag{2.6}$$

The solution is said to be in the implicit form, because the functions $\beta(t)$ and $\gamma(t)$ determined by the boundary conditions (2.2) are not specified yet.

We verify that $\lim_{x\to\infty} u(x,t) = 0$, no matter what β and γ are, as long as they are bounded function of t. Indeed, by the Lebesgue's Dominated Convergence Theorem, we have

$$\int_0^\infty G_t(x + V_0 t - y) u_0(y) dy \to 0 \quad \text{as} \quad x \to \infty$$

if $u_0 \in L^1(\mathbb{R})$, because $G_t(x) \to 0$ as $x \to \infty$. On the other hand, the other three convolution integrals are bounded if $\beta, \gamma \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ and $t > 0$ is finite, because G_t, G'_t, and G''_t have integrable

singularities at $t = 0$. By the same Lebesgue's Dominated Convergence Theorem, these three integrals decay to zero as $x \to \infty$.

The functions $\beta(t)$ and $\gamma(t)$ are to be found from the integral equations obtained at the boundary conditions $u(0,t) = 0$ and $u_x(0,t) = \beta(t)$. These derivations are performed separately for the cases of $V_0 = 0$ and $V_0 \neq 0$.

2.1 Case $V_0 = 0$

We rewrite the solution (2.6) for $V_0 = 0$:

$$u(x,t) = \int_0^\infty G_t(x-y)u_0(y)dy - \frac{1}{2}\int_0^t G'_{t-s}(x)ds + \int_0^t \left[G_{t-s}(x)\gamma(s) + G''_{t-s}(x)\beta(s)\right] ds. \quad (2.7)$$

Using the boundary values (A.3) and (A.4) for the Greens function $G_t(x)$ and the boundary condition $u(0,t) = 0$, we evaluate this expression at $x = 0$ and obtain an integral equation for β and γ:

$$-\frac{1}{4\pi}\Gamma\left(\frac{1}{4}\right)\int_0^t \frac{\gamma(s)}{(t-s)^{1/4}}ds + \frac{1}{4\pi}\Gamma\left(\frac{3}{4}\right)\int_0^t \frac{\beta(s)}{(t-s)^{3/4}}ds \;=\; \int_0^\infty G_t(-y)u_0(y)dy. \quad (2.8)$$

To use the boundary condition $u_x(0,t) = \beta(t)$, we shall recall from equation (A.5) that the function $G'''_t(x)$ behaves like $\mathcal{O}(t^{-1})$ for any $x > 0$ and hence is not integrable in t at $t = 0$. Therefore, we have to be careful to differentiate the solution in the above convolution form. The last term of the solution (2.7) can be computed by using the Fourier transform:

$$v(x,t) := \int_0^t G''_{t-s}(x)\beta(s)ds = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(ik)^2 e^{ikx}\left(\int_0^t e^{-k^4(t-s)}\beta(s)ds\right)dk.$$

Differentiating this expression in x and integrating by parts in s, we obtain

$$\begin{aligned}
v_x(x,t) &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}(ik)^3 e^{ikx}\left(\int_0^t e^{-k^4(t-s)}\beta(s)ds\right)dp \\
&= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{ikx}}{k}\left(\int_0^t \frac{d}{ds}\left(e^{-k^4(t-s)}\right)\beta(s)ds\right)dp \\
&= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{ikx}}{k}\left(\beta(t) - \beta(0)e^{-k^4 t} - \int_0^t e^{-k^4(t-s)}\beta'(s)ds\right)dp \\
&= \frac{1}{2}\beta(t) - \beta(0)H_t(x) - \int_0^t H_{t-s}(x)\beta'(s)ds. \qquad (2.9)
\end{aligned}$$

where

$$H_t(x) := \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{-tk^4+ikx}}{k}dk = \frac{1}{\pi}\int_0^\infty \frac{e^{-tk^4}\sin(kx)}{k}dk = \int_0^x G_t(y)dy. \quad (2.10)$$

Here we note that all integrals are evaluated in the principal value sense, because the half-residue at $k = 0$ is canceled out in the resulting expression (2.9). Also we note that the decay of v_x to

zero as $x \to \infty$ is satisfied because of the symmetry and normalization of G_t in (A.6). We can now use the boundary conditions (A.4) and $u_x(0,t) = \beta(t)$ to obtain the exact value for $\beta(t)$:

$$\begin{aligned}\beta(t) &= 2\int_0^\infty G_t'(-y)u_0(y)dy - \int_0^t G_{t-s}''(0)ds \\ &= 2\int_0^\infty G_t'(-y)u_0(y)dy + \frac{\Gamma(3/4)}{\pi}t^{1/4}. \qquad (2.11)\end{aligned}$$

After $\beta(t)$ is found uniquely from (2.11), $\gamma(t)$ is found uniquely from the integral equation (2.8). This computation completes the construction of the exact solution of the boundary–value problem (1.6) for $V_0 = 0$. Now we turn to the analysis of obtained solution.

Theorem 1 *Consider the advection–diffusion equation (1.6) for $V_0 = 0$ with the initial data $u_0 \in L^1(\mathbb{R}_+)$. Then, there exists a solution $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ of the evolution problem in the explicit form (2.7), where $\beta, \gamma \in L^\infty_{\text{loc}}(\mathbb{R}_+)$ are defined by (2.8) and (2.11) with* $\lim_{t\to\infty}\beta(t) = +\infty$.

Proof. The convolution integral in the explicit expression (2.11) can be analyzed from the Green's function (A.5). If $u_0 \in L^1(\mathbb{R})$, then

$$\left|\int_0^\infty G_t'(-y)u_0(y)dy\right| \le \frac{\|g'\|_{L^\infty}\|u_0\|_{L^1}}{t^{1/2}}, \quad t > 0.$$

Therefore, $\beta \in L^\infty_{\text{loc}}(\mathbb{R}_+)$ and $\beta(t) \sim t^{1/4}$ as $t \to \infty$ due to the second term in (2.11). Now, the integral equation (2.8) for $\gamma(t)$ with a weakly singular kernel is well defined and solutions exist with $\gamma \in L^\infty_{\text{loc}}(\mathbb{R}_+)$. Similarly, the solution $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ is well defined by (2.7). □

Remark 1 *One can show that there is no singularity of the solution for $\beta(t)$ as $t \to 0$ so that $\beta(0) = u_0'(0)$ by continuity. Also, one can show that the solution of the integral equation (2.8) for $\gamma(t)$ exists in the closed form: $\gamma(t) = 2\int_0^\infty G_t(-y)u_0'''(y)dy$.*

Coming back to the original question, if $u_0(0) = 0$, $u_0'(0) < 0$, and $u_0''(0) = -\frac{1}{2}$, then there is a finite value of $t_0 \in (0,\infty)$ such that $u_x|_{x=0} > 0$ for all $t > t_0$, that is, $h(x,t)$ loses monotonicity at $x = 0$ in a finite time t_0 (recall that $u = h_x$). This dynamical phenomenon occurs because of the inhomogeneous boundary conditions $u_{xx}|_{x=0} = -\frac{1}{2}$ even in the absence of the advection term in the fourth-order diffusion equation (1.6).

2.2 Case $V_0 \neq 0$

We have the solution in the implicit form (2.6) and we need to derive integral equations on the unknown function $\beta(t)$ and $\gamma(t)$. One integral equation follows again from the boundary condition $u(0,t) = 0$:

$$\begin{aligned}&-\int_0^t \left[G_{t-s}(V_0(t-s))\gamma(s) + G_{t-s}''(V_0(t-s))\beta(s)\right]ds \\ &= \int_0^\infty G_t(V_0t - y)u_0(y)dy - \frac{1}{2}\int_0^t G_{t-s}'(V_0(t-s))ds. \qquad (2.12)\end{aligned}$$

To find another integral equation from the boundary condition $u_x(0,t) = \beta(t)$, we have to use the technique explained in Section 2.1 and to compute the derivative of the solution (2.6) in x:

$$\begin{aligned} u_x(x,t) &= \int_0^\infty G_t'(x+V_0t-y)u_0(y)dy - \frac{1}{2}\int_0^t G_{t-s}''(x+V_0(t-s))ds \\ &+ \int_0^t G_{t-s}'(x+V_0(t-s))\gamma(s)ds + \frac{1}{2}\beta(t) - \beta(0)H_t(x+V_0t) \\ &- \int_0^t H_{t-s}(x+V_0(t-s))\beta'(s)ds + V_0\int_0^t G_{t-s}(x+V_0(t-s))\beta(s)ds. \end{aligned} \quad (2.13)$$

We can now use the boundary condition $u_x(0,t) = \beta(t)$ to obtain another integral equation for β and γ:

$$\begin{aligned} &\beta(t) + 2\beta(0)H_t(V_0t) + 2\int_0^t H_{t-s}(V_0(t-s))\beta'(s)ds \\ &-2V_0\int_0^t G_{t-s}(V_0(t-s))\beta(s)ds - 2\int_0^t G_{t-s}'(V_0(t-s))\gamma(s)ds \\ &= 2\int_0^\infty G_t'(V_0t-y)u_0(y)dy - \int_0^t G_{t-s}''(V_0(t-s))ds. \end{aligned} \quad (2.14)$$

The system of integral equations (2.12) and (2.14) completes the solution (2.6) for the case $V_0 \neq 0$. Because of the original motivation to study behavior for large negative $V(t)$ in (1.3), we shall analyze the obtained solution for $V_0 < 0$.

Theorem 2 *Consider the advection–diffusion equation (1.6) for $V_0 < 0$ with the initial data $u_0 \in L^1(\mathbb{R}_+)$. Then, there exists a solution $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ of the evolution problem in the explicit form (2.6), where $\beta, \gamma \in L^\infty(\mathbb{R}_+)$ are defined by (2.12) and (2.14) with*

$$\lim_{t\to\infty}\beta(t) = \frac{1}{2|V_0|^{1/3}} \qquad \lim_{t\to\infty}\gamma(t) = \frac{|V_0|^{1/3}}{2}. \quad (2.15)$$

Proof. Similarly to the proof of Theorem 1, it is easy to show from the integral equations (2.12) and (2.14) that if $u_0 \in L^1(\mathbb{R}_+)$, then $\beta, \gamma \in L^\infty_{\text{loc}}(\mathbb{R}_+)$. We shall now compute the limit of $\beta(t)$ and $\gamma(t)$ as $t \to \infty$:

$$\beta_\infty := \lim_{t\to\infty}\beta(t), \quad \gamma_\infty := \lim_{t\to\infty}\gamma(t). \quad (2.16)$$

To deal with the first integral equation (2.12), we first notice the explicit computation by using the Fourier transform:

$$\begin{aligned} f(t) &:= \int_0^t G_{t-s}'(V_0(t-s))ds \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty (ik)\left(\int_0^t e^{-s(k^4-ikV_0)}ds\right)dk \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \frac{i(1-e^{-t(k^4-ikV_0)})}{k^3-iV_0}dk, \end{aligned}$$

where the integrals in s and k can be interchanged by Fubini's Theorem and the integration is performed in the principal value sense. We can now explicitly compute the limit as $t \to \infty$ by using Lebesgue's Dominated Convergence Theorem:

$$\lim_{t\to\infty} f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i}{k^3 - iV_0}dk = \frac{-V_0}{\pi}\int_0^{\infty}\frac{dk}{k^6+V_0^2} = \frac{1}{3|V_0|^{2/3}}.$$

This computation gives the last term of the integral equation (2.12) as $t \to \infty$. To deal with the first term on the right-hand side of (2.12), we write

$$\begin{aligned}\int_0^{\infty} G_t(V_0 t - y)u_0(y)dy &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^{\infty} e^{-t(k^4 - ikV_0) - iky}u_0(y)dy\right)dk\\ &= \int_{-\infty} e^{-t(k^4-ikV_0)}\hat{u}_0(k)dk,\end{aligned}$$

where

$$\hat{u}_0(k) := \frac{1}{2\pi}\int_0^{\infty} e^{-iky}u_0(y)dy.$$

By Lebesgue's Dominated Convergence Theorem, this integral converges to zero as $t \to \infty$ as long as $u_0 \in L^1(\mathbb{R}_+)$.

To deal with the second term on the left-hand side of the integral equation (2.12), we rewrite it in the form

$$\int_0^t G''_{t-s}(V_0(t-s))\beta(s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}(ik)^2\left(\int_0^t \beta(t-s)e^{-s(k^4-ikV_0)}ds\right)dk.$$

Since $\beta \in L^{\infty}_{\rm loc}(\mathbb{R}_+)$ with the assumed limit in (2.16), we apply Lebesgue's Dominated Convergence Theorem and compute the integrals in the principal value sense:

$$\lim_{t\to\infty}\int_0^t G''_{t-s}(V_0(t-s))\beta(s)ds = \frac{-\beta_\infty}{2\pi}\int_{-\infty}^{\infty}\frac{k}{k^3-iV_0}dk = \frac{-\beta_\infty}{\pi}\int_0^{\infty}\frac{k^4 dk}{k^6+V_0^2} = \frac{-\beta_\infty}{3|V_0|^{1/3}}.$$

The first term on the left-hand side of the integral equation (2.12) is more tricky. First, we rewrite it in the form,

$$\int_0^t G_{t-s}(V_0(t-s))\gamma(s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^t \gamma(t-s)e^{-s(k^4-ikV_0)}ds\right)dk.$$

However, if $\gamma \in L^{\infty}_{\rm loc}(\mathbb{R}_+)$ with the limit in (2.16), application of Lebesgue's Dominated Convergence Theorem yields the integral in k with a simple pole at $k = 0$:

$$\lim_{t\to\infty}\int_0^t G_{t-s}(V_0(t-s))\gamma(s)ds = \frac{\gamma_\infty}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{k(k^3-iV_0)}.$$

The integral is no longer understood in the principal value sense. Instead, we return back to the treatment of the inverse Laplace transform in (2.5) with $\mathrm{Re}(c) > 0$, use transformation $o = ik$,

and shift the contour of integration in k below the pole at $k = 0$. As a result, computations are completed with the half-residue term at the simple pole and the principal value integral:

$$\lim_{t\to\infty} \int_0^t G_{t-s}(V_0(t-s))\gamma(s)ds = \frac{\gamma_\infty}{2\pi}\left(\frac{\pi}{|V_0|} + \int_{-\infty}^{\infty} \frac{k^2 dk}{k^6 + V_0^2}\right) = \frac{2\gamma_\infty}{3|V_0|}.$$

Combining all computations together, we have obtained the following linear equation on β_∞ and γ_∞ from the integral equation (2.12):

$$\frac{2\gamma_\infty}{|V_0|} - \frac{\beta_\infty}{|V_0|^{1/3}} = \frac{1}{2|V_0|^{2/3}}. \tag{2.17}$$

To deal with the second integral equation (2.14), we use the Fourier transform again to write

$$H_t(V_0 t) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{-t(k^4 - ikV_0)}}{k} dk$$

and

$$\int_0^t H_{t-s}(V_0(t-s))\beta'(s)ds = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{k}\left(\int_0^t \beta'(t-s)e^{-s(k^4 - ikV_0)}ds\right)dk,$$

where the integrals are understood in the principal value sense. If $\beta, \gamma \in L^\infty_{\rm loc}$ with the limits (2.16), the Lebesgue's Dominated Convergence Theorem implies that

$$H_t(V_0 t), \int_0^t H_{t-s}(V_0(t-s))\beta'(s)ds \to 0 \quad \text{as} \quad t \to \infty.$$

Similar to the previous computations, we prove that

$$\int_0^\infty G_t'(V_0 t - y)u_0(y)dy \to 0 \quad \text{as} \quad t \to \infty,$$

$$\lim_{t\to\infty}\int_0^t G''_{t-s}(V_0(t-s))ds = \frac{-1}{3|V_0|^{1/3}},$$

$$\lim_{t\to\infty}\int_0^t G'_{t-s}(V_0(t-s))\gamma(s)ds = \frac{\gamma_\infty}{3|V_0|^{2/3}},$$

and

$$\lim_{t\to\infty}\int_0^t G_{t-s}(V_0(t-s))\beta(s)ds = \frac{\beta_\infty}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{k(k^3 - iV_0)} = \frac{\beta_\infty}{6|V_0|}.$$

where the last integral is computed in the principal value sense because equations (2.13) and (2.14) are derived in the principal value sense.

Combining all computations together, we have obtained the following linear equation on β_∞ and γ_∞ from the integral equation (2.14):

$$4\beta_\infty - \frac{2\gamma_\infty}{|V_0|^{2/3}} = \frac{1}{|V_0|^{1/3}}. \tag{2.18}$$

Solving the linear system (2.17) and (2.18), we obtain (2.15) and the theorem is proved. $\square$

Coming back to the original question, if $u_0(0) = 0$, $u_0'(0) < 0$, and $u_0'' = -\frac{1}{2}$, then there is a finite value of $t_0 \in (0, \infty)$ such that $u_x|_{x=0} > 0$ for all $t > t_0$. Therefore, like in the case $V_0 = 0$, the function $h(x,t)$ loses monotonicity at $x = 0$ in a finite time t_0 (where $u = h_x$) with the only difference that $u_x|_{x=0}$ remains finite and positive as $t \to \infty$. We conclude that the presence of the advection term with $V_0 < 0$ in the fourth-order diffusion equation (1.6) does not prevent the loss of monotonicity in x but still stabilizes the solution globally as $t \to \infty$. In both cases $V_0 = 0$ and $V_0 < 0$, the monotonicity of h in x is lost because of the inhomogeneous boundary conditions $h_{xx}|_{x=0} = -\frac{1}{2}$.

3 Solution of the original problem

We shall now use Laplace transform (2.1) to obtain the implicit solution to the advection-diffusion equation (1.4) with a variable speed $V(t)$. Let us denote

$$W(t) = \int_0^t V(s)ds$$

and obtain the Laplace transform solution in the form:

$$\begin{aligned} U(p,t) &= U_0(p)e^{-tp^4 + W(t)p} \\ &\quad + \int_0^t e^{-(t-s)p^4 + (W(t)-W(s))p}\left(-\frac{1}{2}p + p^2\beta(s)\right)ds. \end{aligned} \tag{3.1}$$

Compared with the solution (2.4), we have set $\gamma(t) = 0$ because of the third boundary condition in (1.5). Using the inverse Laplace transform in x and recalling the definition of the Green's function $G_t(x)$ (see Appendix A), we obtain the analogue of the implicit solution (2.6):

$$\begin{aligned} u(x,t) &= \int_0^\infty G_t(x + W(t) - y)u_0(y)dy - \frac{1}{2}\int_0^t G_{t-s}'(x + W(t) - W(s))ds \\ &\quad + \int_0^t G_{t-s}''(x + W(t) - W(s))\beta(s)ds. \end{aligned} \tag{3.2}$$

Now we have two unknowns β and W and we can set up two integral equations at the boundary conditions $u(0,t) = 0$ and $u_x(0,t) = \beta(t)$.

From the boundary condition $u(0,t) = 0$, we obtain the integral equation:

$$\begin{aligned} -\int_0^t G_{t-s}''(W(t) - W(s))\beta(s)ds &= \int_0^\infty G_t(W(t) - y)u_0(y)dy \\ &\quad - \frac{1}{2}\int_0^t G_{t-s}'(W(t) - W(s))ds. \end{aligned} \tag{3.3}$$

To find another integral equation from the boundary condition $u_x(0,t) = \beta(t)$, we differentiate the solution (3.2) in x:

$$
\begin{aligned}
u_x(x,t) \quad = \quad & \int_0^\infty G_t'(x+W(t)-y)u_0(y)dy - \frac{1}{2}\int_0^t G_{t-s}''(x+W(t)-W(s))ds \\
& +\frac{1}{2}\beta(t) - \beta(0)H_t(x+W(t)) - \int_0^t H_{t-s}(x+W(t)-W(s))\beta'(s)ds \\
& +V(t)\int_0^t G_{t-s}(x+W(t)-W(s))\beta(s)ds. \qquad (3.4)
\end{aligned}
$$

From the boundary condition $u_x(0,t) = \beta(t)$, we obtain another integral equation:

$$
\begin{aligned}
& \beta(t) + 2\beta(0)H_t(W(t)) + 2\int_0^t H_{t-s}(W(t)-W(s))\beta'(s)ds \\
& -2V(t)\int_0^t G_{t-s}(W(t)-W(s))\beta(s)ds \\
& = 2\int_0^\infty G_t'(W(t)-y)u_0(y)dy - \int_0^t G_{t-s}''(W(t)-W(s))ds. \qquad (3.5)
\end{aligned}
$$

We shall prove that the system of two integral equations (3.3) and (3.5) determines uniquely the function $\beta(t)$ and $V(t)$ locally for $t > 0$. The following theorem gives the result.

Theorem 3 *Assume that $u_0 \in C^\infty(\mathbb{R}_+)$ such that*

$$
u_0(0) = 0, \quad u_0''(0) = -\frac{1}{2}, \quad u_0'''(0) = 0. \qquad (3.6)
$$

Then, there exists a formal solution (V,β) of the system of two integral equations (3.3) and (3.5) in the form of the fractional power series:

$$
\beta(t) = \beta_0 + \sum_{n=4}^\infty \beta_{n/4}t^{n/4}, \quad V(t) = V_0 + \sum_{n=1}^\infty V_{n/4}t^{n/4}, \qquad (3.7)
$$

where $\beta_0 = u_0'(0)$, $V_0 = u_0^{(4)}(0)/u_0'(0)$, and $\{\beta_{n/4}, V_{(n-3)/4}\}_{n=4}^\infty$ are uniquely determined.

Proof. We substitute the series representations (3.7) to each term of the integral equations (3.3) and (3.5). It follows from (3.7) that

$$
a_t = \frac{1}{t^{1/4}}\int_0^t V(s)ds = V_0 t^{3/4} + \sum_{n=1}^\infty \frac{4}{n+4}V_{n/4}t^{(n+3)/4}
$$

and

$$
\xi_{t,\tau} = \frac{1}{\tau^{1/4}}\int_{t-\tau}^t V(s)ds = V_0\tau^{3/4} + \sum_{n=1}^\infty \frac{4}{n+4}V_{n/4}\frac{t^{(n+4)/4}-(t-\tau)^{(n+4)/4}}{\tau^{1/4}}.
$$

Using the representation (A.5) of the Green function with $g \in C^\infty(\mathbb{R})$, we obtain for the three terms of the integral equation (3.3):

$$\int_0^t G''_{t-s}(W(t)-W(s))\beta(s)ds = \beta_0 \int_0^t \frac{g''(\xi_{t,\tau})}{\tau^{3/4}} d\tau + \sum_{n=4}^\infty \beta_{n/4} \int_0^t \frac{g''(\xi_{t,\tau})(t-\tau)^{n/4}}{\tau^{3/4}} d\tau$$

$$= 4\beta_0 g''(0) t^{1/4} + \sum_{k=2}^\infty \frac{1}{k!} g^{(k+2)}(0) \int_0^t \frac{\xi_{t,\tau}^k}{\tau^{3/4}} d\tau + \sum_{n=4}^\infty \beta_{n/4} \int_0^t \frac{g''(\xi_{t,\tau})(t-\tau)^{n/4}}{\tau^{3/4}} d\tau,$$

$$\int_0^\infty G_t(W(t)-y)u_0(y)dy = \int_0^\infty g(z-a_t)u_0(t^{1/4}z)dz = \sum_{n=1}^\infty \frac{1}{n!} u_0^{(n)}(0) t^{n/4} \int_0^\infty g(z-a_t) z^n dz$$

$$= t^{1/4} u_0'(0) \sum_{k=0}^\infty \frac{1}{k!}(-a_t)^k \int_0^\infty g^{(k)}(z) z dz + \sum_{n=2}^\infty \frac{1}{n!} u_0^{(n)}(0) t^{n/4} \int_0^\infty g(z-a_t) z^n dz,$$

and

$$\int_0^t G'_{t-s}(W(t)-W(s))ds = \int_0^t \frac{g'(\xi_{t,\tau})}{\tau^{2/4}} d\tau = \sum_{k=1}^\infty \frac{1}{k!} g^{(k+1)}(0) \int_0^t \frac{\xi_{t,\tau}^k}{\tau^{2/4}} d\tau.$$

At the first powers of $t^{1/4}$, we obtain a system of linear algebraic equations on the coefficients of the power series (3.7):

$$\begin{aligned}
t^{1/4}: &\quad -4\beta_0 g''(0) = u_0'(0) \int_0^\infty g(z) z dz, \\
t^{2/4}: &\quad 0 = \frac{1}{2!} u_0''(0) \int_0^\infty g(z) z^2 dz, \\
t^{3/4}: &\quad 0 = \frac{1}{3!} u_0'''(0) \int_0^\infty g(z) z^3 dz, \\
t^{4/4}: &\quad 0 = \frac{1}{4!} u_0^{(4)}(0) \int_0^\infty g(z) z^4 dz - u_0'(0) V_0 \int_0^\infty g'(z) z dz, \\
t^{5/4}: &\quad -\beta_{4/4} g''(0) \int_0^1 \frac{(1-x)^{4/4}}{x^{3/4}} dx = \frac{1}{5!} u_0^{(5)}(0) \int_0^\infty g(z) z^5 dz - \frac{1}{2} u_0''(0) V_0 \int_0^\infty g'(z) z^2 dz \\
&\qquad - \frac{4}{5} u_0'(0) V_{1/4} \int_0^\infty g'(z) z dz - \frac{2}{5} g''(0) V_0,
\end{aligned}$$

and so on. Using the explicit values for the integrals (A.9)–(A.13) and the initial conditions (3.6), we obtain $\beta_0 = u_0'(0)$, $V_0 = u_0^{(4)}(0)/u_0'(0)$, and the linear equation

$$u_0'(0) V_{1/4} + 8 g''(0) \left(\beta_{4/4} + u_0^{(5)}(0) + \frac{1}{2} V_0 \right) = 0 \tag{3.8}$$

Similarly, we work with the terms of the second integral equation (3.5):

$$\int_0^t G_{t-s}(W(t)-W(s))\beta(s)ds = \beta_0 \int_0^t \frac{g(\xi_{t,\tau})}{\tau^{1/4}}d\tau + \sum_{n=4}^{\infty}\beta_{n/4}\int_0^t \frac{g(\xi_{t,\tau})(t-\tau)^{n/4}}{\tau^{1/4}}d\tau$$

$$= \frac{4}{3}\beta_0 g(0)t^{3/4} + \sum_{k=2}^{\infty}\frac{1}{k!}g^{(k)}(0)\int_0^t \frac{\xi_{t,\tau}^k}{\tau^{1/4}}d\tau + \sum_{n=4}^{\infty}\beta_{n/4}\int_0^t \frac{g(\xi_{t,\tau})(t-\tau)^{n/4}}{\tau^{1/4}}d\tau,$$

$$\int_0^\infty G'_t(W(t)-y)u_0(y)dy = \int_0^\infty g(z-a_t)u'_0(t^{1/4}z)dz = \sum_{n=0}^{\infty}\frac{1}{n!}u_0^{(n+1)}(0)t^{n/4}\int_0^\infty g(z-a_t)z^n dz$$

$$= u'_0(0)\sum_{k=0}^{\infty}\frac{1}{k!}(-a_t)^k\int_0^\infty g^{(k)}(z)dz + \sum_{n=1}^{\infty}\frac{1}{n!}u_0^{(n+1)}(0)t^{n/4}\int_0^\infty g(z-a_t)z^n dz,$$

$$\int_0^t G''_{t-s}(W(t)-W(s))ds = \int_0^t \frac{g''(\xi_{t,\tau})}{\tau^{3/4}}d\tau = \sum_{k=0}^{\infty}\frac{1}{k!}g^{(k+2)}(0)\int_0^t \frac{\xi_{t,\tau}^k}{\tau^{3/4}}d\tau,$$

$$H_t(W(t)) = \int_0^{a_t} g(z)dz = \sum_{k=0}^{\infty}\frac{1}{(k+1)!}g^{(k)}(0)a_t^{k+1},$$

and

$$\int_0^t H_{t-s}(W(t)-W(s))\beta'(s)ds = \sum_{n=4}^{\infty}\beta_{n/4}\sum_{k=0}^{\infty}\frac{1}{(k+1)!}g^{(k)}(0)\int_0^t \xi_{t,\tau}^{k+1}(t-\tau)^{(n-4)/4}d\tau$$

At the first powers of $t^{1/4}$, we obtain a system of linear algebraic equations on the coefficients of the power series (3.7):

$$t^{0/4}: \qquad \beta_0 = 2u'_0(0)\int_0^\infty g(z)dz,$$

$$t^{1/4}: \qquad 0 = 2u''_0(0)\int_0^\infty g(z)zdz - 4g''(0),$$

$$t^{2/4}: \qquad 0 = u'''_0(0)\int_0^\infty g(z)z^2dz,$$

$$t^{3/4}: \qquad 2\beta_0 g(0)V_0 - \frac{8}{3}\beta_0 g(0)V_0 = \frac{1}{3}u_0^{(4)}(0)\int_0^\infty g(z)z^3dz - 2u'_0(0)V_0\int_0^\infty g'(z)dz,$$

$$t^{4/4}: \qquad \beta_{4/4} + \frac{8}{5}\beta_0 g(0)V_{1/4} - \frac{8}{3}\beta_0 g(0)V_{1/4} = \frac{1}{12}u_0^{(5)}(0)\int_0^\infty g(z)z^4dz - 2u''_0(0)V_0\int_0^\infty g'(z)zdz$$
$$-\frac{8}{5}u'_0(0)V_{1/4}\int_0^\infty g'(z)dz,$$

and so on. Again, using the explicit values for the integrals (A.9)–(A.13) and the initial conditions (3.6), we obtain $\beta_0 = u_0'(0)$, $V_0 = u_0^{(4)}(0)/u_0'(0)$, and the linear equation

$$-\frac{8}{3}u_0'(0)g(0)V_{1/4} + \left(\beta_{4/4} + u_0^{(5)}(0) + \frac{1}{2}V_0\right) = 0 \tag{3.9}$$

The system of linear equations (3.8) and (3.9) has a unique solution

$$V_{1/4} = 0, \quad \beta_{4/4} = -u_0^{(5)}(0) - \frac{1}{2}V_0, \tag{3.10}$$

provided that

$$-\frac{64}{3}g(0)g''(0) = \frac{4}{3\pi^2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{4\sqrt{2}}{3\pi} \neq 1,$$

which is confirmed. Note that the constraint $V_0 = u_0^{(4)}(0)/u_0'(0)$ also follows from the contact equation (1.2) obtained for sufficiently smooth solutions. Similarly, the second equation (3.10) follows from the advection–diffusion equation (1.4) after one derivative in x is taken in the limit $x \to 0$ and $t \to 0$.

It remains to prove that the system of linear equations obtained from the system of integral equations (3.3) and (3.5) can be solved at each order of $t^{n/4}$ for $n \in \mathbb{N}$. From the previous computations, we can deduce that the first integral equation at $t^{(n+1)/4}$ gives a linear equation on variables $(\beta_{n/4}, V_{(n-3)/4})$ of the power series (3.7):

$$-\beta_{n/4}g''(0)\int_0^1 \frac{(1-x)^{n/4}}{x^{3/4}}dx + \frac{4}{n+1}u_0'(0)V_{(n-3)/4}\int_0^\infty g'(z)zdz = \cdots, \tag{3.11}$$

where the dots denote the terms expressed through derivatives of $u_0(x)$ at $x = 0$ and the previous terms of the power series (3.7). Similarly, the second integral equation at $t^{n/4}$ gives another linear equation on variables $(\beta_{n/4}, V_{(n-3)/4})$:

$$\beta_{n/4} - \frac{8}{3}\beta_0 g(0)V_{(n-3)/4} = \cdots. \tag{3.12}$$

The system of linear equations (3.11) and (3.12) is non-degenerate if

$$C_n := -\frac{4(n+1)}{3}g(0)g''(0)\int_0^1 \frac{(1-x)^{n/4}}{x^{3/4}}dx = \frac{(n+1)}{6\sqrt{2}\pi}\frac{\Gamma\left(\frac{n+4}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{n+5}{4}\right)} \neq 1, \quad n \in \mathbb{N}. \tag{3.13}$$

The left-hand side $\{C_n\}_{n\in\mathbb{N}}$ is computed numerically (see Figure 2). It is a monotonically increasing sequence that approaches closely to 1 at $n = 8$, where $C_8 \approx 0.96$, and $n = 9$, where $C_9 \approx 1.04$. Therefore, the linear system is non-degenerate and a unique solution for $(\beta_{n/4}, V_{(n-3)/4})$ exists for any $n \in \mathbb{N}$. □

In the present time, we cannot prove yet that the system of integral equations (3.3) and (3.5) leads to a finite-time blow-up, according to the conjecture in [1]. Nevertheless, numerical computations show that the blow-up holds for a generic set of initial data. Figure 3 shows the

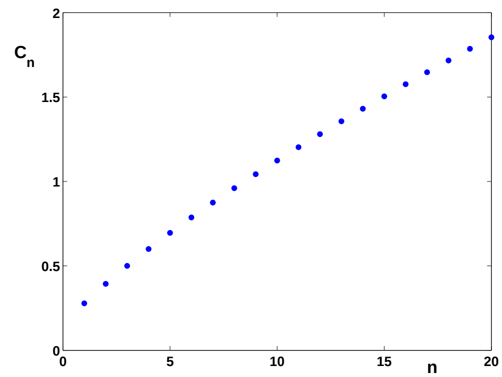

Figure 2: Numerical approximations of C_n defined by (3.13).

behavior of functions $\beta(t)$ and $V(t)$ near the blow-up time. It follows from this figure that $\beta(t) = h_{xx}|_{x=0} \to 0$ at the same time as $V(t) \to -\infty$ with $\beta(t)V(t)^{1/3} \to C_0$, where $C_0 > 0$ is a numerical constant. In other words, we conclude with the conjecture that $\beta(t) \sim V(t)^{-1/3}$ as $V(t) \to -\infty$ in a finite time $t_0 \in (0, \infty)$.

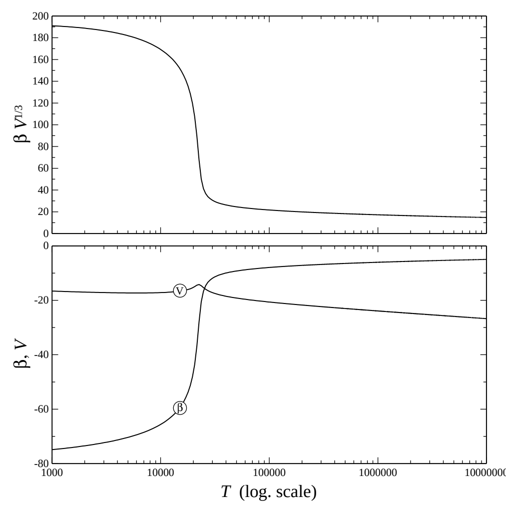

Figure 3: Numerical computations of $\beta(t)$ and $V(t)$ for the advection-diffusion equation (1.1). We thank the authors of [1] for this numerical figure.

A Green's function

Let us define the fundamental solution of the fourth-order diffusion equation:

$$\begin{cases} h_t + h_{xxxx} = 0, & x \in \mathbb{R}, \ \ t > 0, \\ h|_{t=0} = \delta(x), & x \in \mathbb{R}, \end{cases} \tag{A.1}$$

where δ is a standard Dirac delta-function in the distribution sense. The fundamental solution is usually referred to as Green's function and we shall denote it by

$$h(x,t) = G_t(x), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+.$$

Using the Fourier transform in x, we can obtain the explicit expression for Green's function:

$$G_t(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-tk^4 + ikx} dk = \frac{1}{\pi} \int_0^{\infty} e^{-tk^4} \cos(kx) dk. \tag{A.2}$$

In particular, we have $G_t(-x) = G_t(x)$ for all $x \in \mathbb{R}$ and

$$G_t(0) = \frac{1}{\pi} \int_0^{\infty} e^{-tk^4} dk = \frac{\Gamma(1/4)}{4\pi t^{1/4}}, \tag{A.3}$$

$$G_t''(0) = -\frac{1}{\pi} \int_0^{\infty} k^2 e^{-tk^4} dk = -\frac{\Gamma(3/4)}{4\pi t^{3/4}}, \tag{A.4}$$

where Γ is the standard Gamma function. The Green's function can be represented in the self-similar form by

$$G_t(x) = \frac{1}{t^{1/4}} g\left(\frac{x}{t^{1/4}}\right), \quad g(z) = \frac{1}{\pi} \int_0^{\infty} e^{-k^4} \cos(kz) dk, \tag{A.5}$$

where $g \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Therefore, G_t decays to zero as $t \to \infty$ in any L^p norm for $p \geq 2$. In particular, $|G_t(x)| \leq \|g\|_{L^\infty}/t^{1/4}$, $|G_t'(x)| \leq \|g'\|_{L^\infty}/t^{1/2}$, and so on, for any $x \in \mathbb{R}$.

By the stationary phase method (see, e.g., Chapter 5 in [3]), $g(z)$ and all derivatives of $g(z)$ decay to zero as $|z| \to \infty$ faster than any algebraic powers. This gives the decay of $G_t(x)$ and any x-derivative of $G_t(x)$ as $|x| \to \infty$ for any fixed $t > 0$. Although G_t and g are not L^1 functions, they satisfy the normalization conditions:

$$\int_{\mathbb{R}} G_t(x) dx = \int_{\mathbb{R}} g(z) dz = 1, \quad t > 0. \tag{A.6}$$

The even function $g : \mathbb{R} \to \mathbb{R}$ satisfies the ordinary differential equation

$$4 \frac{d^4 g}{dz^4} = g + z \frac{dg}{dz}, \quad z \in \mathbb{R}, \tag{A.7}$$

subject to the initial values

$$g(0) = \frac{1}{4\pi} \Gamma\left(\frac{1}{4}\right), \quad g'(0) = 0, \quad g''(0) = -\frac{1}{4\pi} \Gamma\left(\frac{3}{4}\right), \quad g'''(0) = 0, \tag{A.8}$$

and the decay behavior as $|z| \to \infty$. It is clear from the differential equation that $g \in C^\infty(\mathbb{R})$ satisfies a number of integral constraints:

$$\int_0^\infty zg(z)dz = -4g''(0), \tag{A.9}$$

$$\int_0^\infty z^2g(z)dz = 0, \tag{A.10}$$

$$\int_0^\infty z^3g(z)dz = -8g(0), \tag{A.11}$$

$$\int_0^\infty z^4g(z)dz = -12, \tag{A.12}$$

$$\int_0^\infty z^5g(z)dz = 164!g''(0), \tag{A.13}$$

and so on.

References

[1] E.S. Benilov and M. Vynnycky, "Contact lines with a 180° contact angle", submitted to J. Fluid Mech. (2012)

[2] D.J. Benney and W.J. Timson, "The rolling motion of a viscous fluid on and off a rigid surface", Stud. Appl. Math. **63** (1980), 93–98.

[3] P. Miller, *Applied Asymptotic Analysis*, Graduate Studies in Mathematics **75** (AMS Publications, Providence, 2006).

[4] C.G. Ngan and V.E.B. Dussan, "The moving contact line with a 180° advancing contact angle", Phys. Fluids **24** (1984), 2785–2787.